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Dr. Chun's Numb3rs & L⌀gic

Stable Marriage Problem

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Stable Marriage Problem

Alan, Bob, Carl and Dan, four bachelors in Dr. Chun's class, finally contemplate marriage. They approach Dr. Chun, the matchmaker, who introduces them to Alice, Brenda, Cindy and Debbie. After the meeting, each person ranks all of the members of the opposite sex, and hands it to Dr. Chun.

(a) Men's preference lists

	Alice	Brenda	Cindy	Debbie
Alan	2	4	1	3
Bob	1	4	2	3
Carl	3	4	2	1
Dan	3	2	1	4

(b) Women's preference lists

	Alice	**Brenda**	**Cindy**	**Debbie**
Alan	3	3	3	4
Bob	4	4	2	2
Carl	2	1	1	3
Dan	1	2	4	1

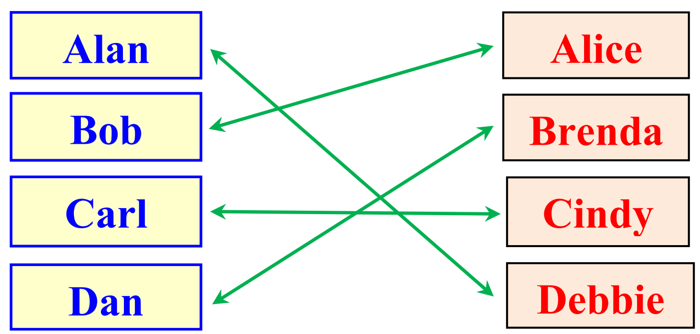
Dr. Chun's job is to find a match for each man, and his reputation depends on the number of *successful* marriages arranged.

What should Dr. Chun do?

* Stable Marriages

If man *i* and woman *j* have not been paired up, then at least one of them should be paired with a "better" partner. In other words, if *i* and *j* are not married to each other, either *i* is married to somebody he prefers to *j*, or *j* is married to someone she prefers to *i*.

If this is not true, then man *i* and woman *j* will find it beneficial to divorce their respective spouses and marry each other. Such an arranged pair (*i*, *j*) is called unstable in the match making problem.



The "stable marriage problem" is the problem of finding a stable matching between two equally sized sets of elements given an ordering of preferences for each element. A matching is a mapping from the elements of one set to the elements of the other set.

* Unstable Marriages

▪ Example 1. Two couples

Consider, for example, the following two couples. Suppose that Dr. Chun matches (Alan & Alice) and (Bob & Brenda).

	Alice	Brenda
Alan	2	1
Bob	1	2

	Alice	Brenda
Alan	2	1
Bob	1	2

Observe that Alan likes Brenda better than Alice (his current wife), and Alice likes Bob better than Alan (her current husband); so, the arranged marriage (Alan & Alice) is unstable and would "break-up".

Likewise, Bob likes Alice better than Brenda (his current wife), and Alice likes Bob better than Alan (her current husband); so, Bob and Alice will find it beneficial to divorce their respective spouses and would "elope".

- **Example 2. Four couples**

Suppose that there are four couples and Dr. Chun matches (Alice & Alan), (Brenda & Bob), (Cindy & Carl), and (Debbie & Dan) haphazardly.

	Alice	Brenda	Cindy	Debbie
Alan	2	4	1	3
Bob	1	4	2	3
Carl	3	4	2	1
Dan	3	2	1	4

	Alice	**Brenda**	**Cindy**	**Debbie**
Alan	3	3	3	4
Bob	4	4	2	2
Carl	2	1	1	3
Dan	1	2	4	1

Observe that Dan likes Brenda better than Debbie (his current wife), and Brenda likes Dan better than Bob (her current husband).

Also observe that Dan also likes Alice better than Debbie (his current wife), and Alice likes Dan better than Alan (her current husband). So, the arranged marriage (Dan & Debbie) is definitely unstable.

* **Solution**

Obviously, unstable marriages are undesirable for Dr. Chun, and so the least he should look for is a stable marriage for every couple, where no pair of man and woman will find it beneficial to divorce their respective spouses and marry each other.

- Can Dr. Chun always find a stable marriage for every couple?
- How many stable marriages are there?
- Are some stable marriages better than the others?
- Can people misrepresent their true preferences and thereby gain an advantage?

In 1962, David Gale and Lloyd Shapley proved that, for any equal number of men and women, it is always possible to solve the match making problem and make all marriages *stable*. They presented an efficient algorithm to do so.

We can also formulate the stable marriage problem as an optimization model and answer some of these questions!

Optimization Model

* Integer programming

- **Decision variables**

We have 16 zero-one binary variables.

	Alice	Brenda	Cindy	Debbie
Alan	x_{11}	x_{12}	x_{13}	x_{14}
Bob	x_{21}	x_{22}	x_{23}	x_{24}
Carl	x_{31}	x_{32}	x_{33}	x_{34}
Dan	x_{41}	x_{42}	x_{43}	x_{44}

X_{ij} = 1 if i and j are paired
0 Otherwise

- **Matching constraints**

Each person has exactly one partner of the opposite sex.

	Alice	Brenda	Cindy	Debbie	
Alan		1			1
Bob				1	1
Carl	1				1
Dan			1		1
	1	1	1	1	4

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1,2,\ldots,n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1,2,\ldots,n$$

In total, we have 8 matching constraints.

- **Stability constraints**

1. For the pair $X_{11}=1$ (i.e., Alan & Alice)

Any marriage in which (a) Alan is married to either Brenda(X_{12}) or Debbi(X_{14}), and (b) Alice is married to Bob(X_{21}), is unstable.

	Alice	Brenda	Cindy	Debbie
Alan	**2**	**4**	1	**3**
Bob	1	4	2	3
Carl	3	4	2	1
Dan	3	2	1	4

	Alice	Brenda	Cindy	Debbie
Alan	**3**	3	3	4
Bob	**4**	4	2	2
Carl	2	1	1	3
Dan	1	2	4	1

Alan and Alice would run away secretly to get married. To prevent such a case, impose the following stability constraint:

$$X_{11} + X_{12} + X_{14} + X_{21} \le 1$$

Suppose $X_{11}=0$. Then it could be unstable if (X_{12} or $X_{14} = 1$) and ($X_{21}=1$).

In total, we have 16 stability constraints.

2. For the pair $X_{12}=1$ (i.e., Alan & Brenda)

Alan 's worst choice is Brenda; Alan and Brenda will never elope!

	Alice	Brenda	Cindy	Debbie
Alan	2	4	1	3
Bob	1	4	2	3
Carl	3	4	2	1
Dan	3	2	1	4

	Alice	Brenda	Cindy	Debbie
Alan	3	3	3	4
Bob	4	4	2	2
Carl	2	1	1	3
Dan	1	2	4	1

$X_{12} + X_{22} \leq 1$

Suppose $X_{12}=0$. Then the constraint is always satisfied.

3. For the pair $X_{13}=1$ (i.e., Alan & Cindy)

	Alice	Brenda	Cindy	Debbie
Alan	2	4	1	3
Bob	1	4	2	3
Carl	3	4	2	1
Dan	3	2	1	4

	Alice	Brenda	Cindy	Debbie
Alan	3	3	3	4
Bob	4	4	2	2
Carl	2	1	1	3
Dan	1	2	4	1

$X_{11} + X_{12} + X_{13} + X_{14} + X_{43} \leq 1$

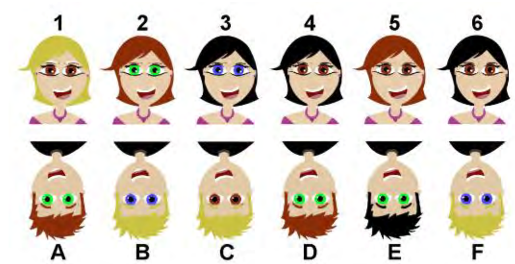
Suppose $X_{13}=0$. It is unstable if (X_{11}, X_{12} or X_{14} = 1) and (X_{43}=1).

Applications

The stable marriage problem has held a fascination for computer scientists, mathematicians, and economists ever since its introduction in 1960's. Research conducted during the past 50 years has helped us understand and appreciate its connections to a variety of problems arising in operations research, combinatorics, and economics.

It has been applied to, for example, the matching of graduating medical students to hospitals, fraternity and sorority rush, college admissions, user matches in a large distributed Internet service, and so on.

In 2012, the Nobel Prize in Economics was awarded to Lloyd S. Shapley and Alvin E. Roth "for the theory of stable allocations and the practice of market design."



> "By all means marry; if you get a good wife, you will be happy. If not, you'll become a philosopher." - Socrates

Movie Trivia

A reporter is assigned to write a story about a woman who has left a string of fiancés at the altar.

Runaway Bride (1999)

A reporter is assigned to write a story about a woman who has left a string of fiancés at the altar.